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14. ABSTRACT
Computational simulation-based predictions are central to science and engineering and to risk assessment and decision making in economics, public policy, and military venues, including several of importance to Air Force missions. Unfortunately, predictions are often fraught with uncertainty so that effective means for quantifying that uncertainty are of paramount importance. The research effort investigates and resolves important algorithmic, mathematical, and practical issues related to the efficient, accurate, and robust computational determination of the quantities of interest used by engineers and decision makers that are determined from solutions of partial differential equations.

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ADVANCED NUMERICAL METHODS FOR COMPUTING STATISTICAL QUANTITIES OF INTEREST FROM SOLUTIONS OF SPDES

FINAL REPORT – FA9550-08-1-0415

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Abstract

Computational simulation-based predictions are central to science and engineering and to risk assessment and decision making in economics, public policy, and military venues, including several of importance to Air Force missions. Unfortunately, predictions are often fraught with uncertainty so that effective means for quantifying that uncertainty are of paramount importance. The research effort investigates and resolves important algorithmic, mathematical, and practical issues related to the efficient, accurate, and robust computational determination of the quantities of interest used by engineers and decision makers that are determined from solutions of partial differential equations having random inputs. Notable accomplishments include the development of a novel approach to discretizing white noise and its application to the stochastic Navier-Stokes-Boussinesq system; development of efficient methods for solving high-dimensional backward stochastic differential equations; verifying the accuracy and effectiveness of approximate deconvolution models on the solution of the stochastic Navier-Stokes equations; error analyses of finite element approximations of the stochastic Stokes equations; error estimates for stochastic optimal Neumann boundary control problems; the development of ANOVA expansions and efficient sampling methods for parameter dependent nonlinear problems; and the analysis of nonlinear spectral eddy-viscosity models of turbulence.

developing finite element methods for parameter identification and control of elliptic equations with random input data; developing stochastic reduced-order models that combine reduced-order models for spatial discretization and efficient stochastic parameter sampling techniques; using ANOVA expansions to study the impact of parameter dependent boundary conditions on solutions of nonlinear partial differential equations and related optimization problems; developing numerical methods for option pricing problems in the presence of random arbitrage return.

1. Novel algorithm for discretizing white noise [10]

The goal of this project is to develop, analyze, and test a novel means for discretizing white noise in the context of partial differential equations. We describe our approach on the simple nonlinear parabolic equation; however, the approach is applicable and effective on more general systems, as is evidenced by the project described in Section 2.

Consider the equation

$$u_t - \kappa \Delta u + u^3 = \sigma dW_t \quad \text{in } (0,T] \times D \tag{1}$$

along with boundary and initial conditions, where the noise term is defined as

$$dW_t(t,x) = \sum_{i=1}^{\infty} w_i \dot{\varsigma}_i(t) \psi_i(x) \quad \text{in } (0,T] \times D \tag{2}$$

and where $\{\varsigma_i(t)\}$ is a set of i.i.d. one-dimensional Brownian motions, the functions $\{\psi_i\}$ form an orthogonal basis of $L^2(D)$ or a subspace, and the coefficients $\{w_i\}$ are to be determined to guarantee the convergence of the series. In particular, we consider the orthogonal basis $\{\psi_i\}$ generated by the elliptic eigenvalue problem $-\kappa\Delta\psi = \lambda\psi$. In (1), σ is a variance parameter and κ a constant.

To treat the white noise (2), we introduce the *Ornstein-Uhlenbeck process* η defined by the stochastic parabolic equation

$$\eta_t - \kappa\Delta\eta = -\alpha\eta + \sigma dW_t. \tag{3}$$

It is not difficult to show that the covariance of η is given by

$$Cov(\eta(t),\eta(s)) = \sigma^2 \sum_{i=1}^{\infty} \frac{w_i^2\psi_i^2}{2(\lambda_i+\alpha)}(e^{-(\lambda_i+\alpha)|t-s|} - e^{-(\lambda_i+\alpha)(t+s)}).$$

Because η is a correlated random field with a known covariance, it can be accurately approximated by a truncated Karhunen-Loeve (KL) expansion.

Now we let $u = v + \eta$. Then, substituting into (1), we have

$$v_t - \kappa\Delta v + (v+\eta)^3 = \alpha\eta \quad \text{in } (0,T]\times D. \tag{4}$$

The advantage of treating (4) instead of (1) is that the latter requires the discretization of white noise whereas, for the former, one need only discretize colored noise having a known covariance function. In fact, if η is a truncated KL expansion, one can apply known efficient methods, including polynomial chaos and stochastic collocation methods, for discretizing (4).

We have proven the *existence* and *uniqueness* of the solutions (4) and also implemented and tested our method. An example of the effectiveness of our approach is given in Section 2.

2. Approximation of the stochastic Navier-Stokes-Boussinesq system [11]

The goal of this project is to consider the effects of noise on fluid-thermal systems. We use the approach described in Section 1 to treat white noise through an auxiliary colored noise process.

We consider the stochastic Navier-Stokes-Boussinesq (NSB) system

$$\begin{aligned} &\mathbf{u}_t - \nu\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \theta\vec{e}_y \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } (0,T]\times D \\ &\theta_t + \mathbf{u}\cdot\nabla\theta = \kappa\triangle\theta + \sigma dW_t \quad \text{in } (0,T]\times D \end{aligned} \tag{5}$$

where the noise term is defined as in (2).

As described in Section 1, we treat the white noise (2) by introducing the Ornstein-Uhlenbeck process η defined by (3). Letting $\theta = \xi + \eta$, the stochastic NSB system is transformed to

$$\begin{aligned} &\mathbf{u}_t - \nu\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = (\xi+\eta)\vec{e}_y \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } (0,T]\times D \\ &\xi_t + \mathbf{u}\cdot\nabla(\xi+\eta) = \kappa\triangle\xi + \eta \quad \text{in } (0,T]\times D. \end{aligned} \tag{6}$$

The advantage of treating (6) instead of (5) is again that the latter requires the discretization of white noise whereas, for the former, one need only discretize colored noise having a known covariance function. In fact, if η is a truncated KL expansion, one can apply known efficient methods, including polynomial chaos and stochastic collocation methods, for discretizing (6).

We have proven the *existence* and *uniqueness* of the solutions (6) and also implemented and tested our method. Figure 1 presents a comparison of the solution of (5) obtained by a Monte-Carlo method and that of (6) using a stochastic collocation method with fewer samples. The difference between the two results is quantified by $E[\|\mathbf{u}_{MC} - \mathbf{u}_{col}\|_{L^2(D)} + \|\theta_{MC} - \theta_{col}\|_{L^2(D)}] = 0.13561$.

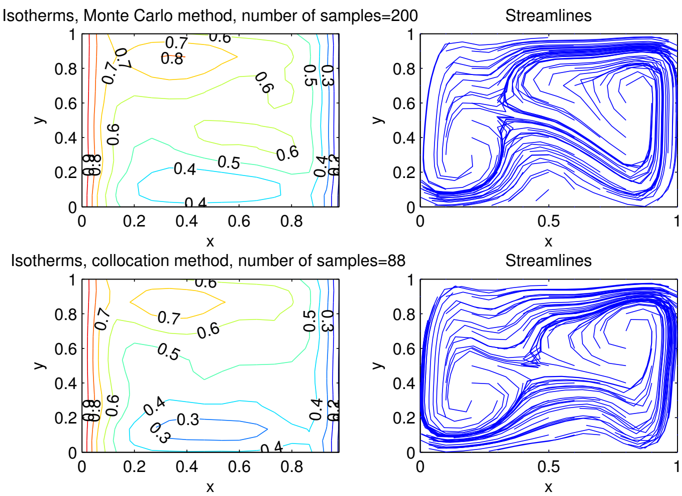


Figure 1: *Comparison of the solution of* (5) *obtained by a Monte-Carlo method (top) and that of* (6) *using a stochastic collocation method with fewer samples (bottom).*

Further computational testing is currently being carried out, including comparisons with results obtained using a larger number of Monte Carlo samples and with polynomial chaos expansions.

3. Solving high-dimensional backward stochastic differential equations [13]

Backward stochastic differential equations (BSDEs) are connected to nonlinear partial differential equations and non-linear semigroups, to the theory of hedging and pricing of contingent claims and, perhaps of most practical interest, to stochastic control problems. This project develops new and more efficient means for discretizing high-dimensional BSDEs.

We consider on the development of efficient numerical methods for solve the abstract BSDE

$$\left\{ \begin{array}{rcl} -dy_t &=& f(t, y_t, z_t)dt - z_t dW_t, \quad t \in [0, T) \\ y_T &=& \xi \end{array} \right. \tag{7}$$

in high (probabilistic parameter) dimensions, using a high-accuracy multi-step method for temporal discretization. In order to make the high-dimensional problems solvable, we mainly focus on two improvements:

- reducing the size of the time-space domain by constructing a new kind of Gauss-Hermite process in order to enhance the efficiency of the multi-step method;
- applying sparse grid techniques so that high-dimensional problems can be solved.

We construct an efficient scheme for solving BSDEs by the defined Gauss-Hermite process. In our scheme, we can truncate the original time-space domain as shown in Figure 2. If the boundaries b is set appropriately, we can just do the computations in the truncated domain without compromising accuracy because, if b is sufficiently large, the neglected parts of the domain are visited with relatively low probability. Note that the reduced time-space domain (lying between $x - b$ and $x + b$) is much smaller than the original one, especially in high-dimensional cases.

We take the following two dimensional example to test the effectiveness and efficiency of our new scheme. Let $W_t = (W_t^1, W_t^2)^T$ denote a two-dimensional Brownian motion, where W_t^1 and W_t^2

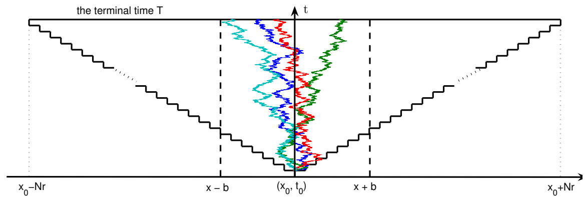


Figure 2: *Truncation of the original time-space domain.*

are two independent, standard one-dimensional Brownian motions. We consider the BSDE

$$\begin{cases} -dy_t = \Big[\dfrac{3}{2}y_t + \ln((z_t\vec{A})^2 + y_t^2) - 2tz_t\vec{A}\Big]dt - z_t dW_t, \\ \quad y_T = \sin(\vec{M}W_T + T^2)\exp(-(\vec{N}W_T)^2) \end{cases}$$

driven by such W_t, where $z_t = (z_t^1, z_t^2)^T$, $\vec{A} = (1,0)^T$, $\vec{M} = (1,0)^T$, and $\vec{N} = (0,1)^T$. Note that here the generator is nonlinear and explicitly depends on t, y_t, and z_t. Eight Guass points are used to approximate the conditional mathematical expectations. The number of time steps are set to $N = 64$, 128, and 256. For the radius b of the truncated spatial domain, we select values between zero and the original radius derived from the Gauss-Hermite quadrature rule in order to test how the errors and computational times vary as the cutoff radius b increases. Errors and computational times are shown in the Figure 3. We conclude that, after using the Gauss-Hermite process in our scheme, the accuracy of the numerical solution y_0^0 and z_0^0 remains the same for any radius $b \geq 8$ but the computational time can increase by almost one order of magnitude as b increases. Therefore, using our scheme, the time-space domain can be reduced significantly leading to a decrease in the number the mesh points without a loss of accuracy. On the other hand, if $b < 8$, accuracy is lost because the neglected portions of the space-time domain are visited with relatively high probability.

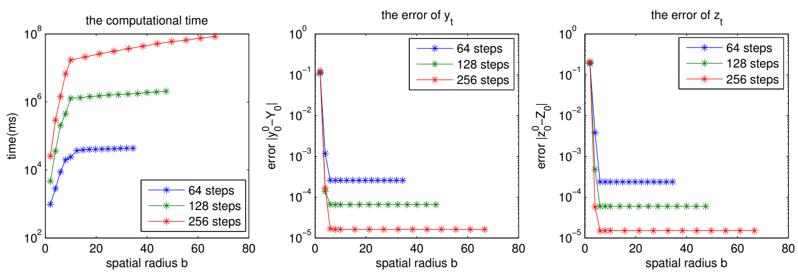


Figure 3: *Errors and computational times using the Gauss-Hermite quadrature process.*

We have also incorporated sparse-sampling and parallelization techniques to further enhance efficiency. We use a hierarchical basis approach to approximation and then have to deal with the fact that these basis functions may not have disjoint support so that we cannot interpolate with respect to each basis function independently; it is required to do the interpolation level by level from the coarser to the finer basis functions. In order to resolve this difficulty, we have to reorganize the order of the nodes of the sparse grid such that the nodes of the same level are grouped together. Then, we implement parallel interpolation within the same level.

4. Effects of approximate deconvolution models on the solution of the stochastic Navier-Stokes equations [5]

Direct numerical simulations of turbulent flows via the Navier-Stokes (NS) equations is not possible because of the onerous grid-size requirements resulting from the need to resolve small-scale behaviors. For this reason, turbulence models are introduced that attempt to account for the effect the small scales have on large-scale behavior without explicitly resolving the small scales. One popular family of turbulence models are approximate deconvolution models.

Fluid systems are subject to noise as a result of which substantial effort has been dedicated to studying stochastic versions of the NS equations. Our goal is to perform a computational study to see what effect the application of turbulence models, and in particular approximate deconvolution models, have on solutions of the stochastic NS equations. To further complicate our setting, we apply noise to the boundary condition data, even though approximate deconvolution have been proven to be accurate only for periodic or zero boundary conditions, even in the deterministic case. We have shown that, for a variety of test problems, the computational results verify the claimed accuracy of the model, even in the probabilistic setting with noise in the boundary data.

We consider random boundary data. The ADM [24] for the stochastic NSE reads

$$w_t - \frac{1}{\mathrm{Re}}\Delta w + \nabla\cdot\overline{(G_N w)(G_N w)}^{\delta} + \nabla q = \overline{f}^{\delta}, \qquad \nabla\cdot w = 0, \qquad w(0,x) = \overline{u}_0^{\delta}(x) \tag{8}$$

and boundary condition with noisy data: $w(t,x,\omega)|_{\partial\Omega} = \overline{u}^{\delta}(t,x)|_{\partial\Omega} + \sum_{i=1}^{K}\omega_i\Phi_i$. Here, G_N is an approximate deconvolution operator defined as follows. For a fixed finite N, define the Nth approximate deconvolution operator G_N by

$$G_N\phi = \sum_{n=0}^{N}(I - A_\delta^{-1})^n\phi,$$

where the averaging operator A_δ^{-1} is the differential filter: given $\phi \in L_0^2(\Omega)$, $\overline{\phi}^{\delta} \in H^2(\Omega)\cap L_0^2(\Omega)$ is the unique solution of

$$A_\delta\overline{\phi}^{\delta} := -\delta^2\Delta\overline{\phi}^{\delta} + \overline{\phi}^{\delta} = \phi \quad \text{in } \Omega.$$

Sample results are provided in Table 1 in which the expected value and variance of the difference between the direct numerical simulation and approximate deconvolution solutions are given for the case of flow past a step; here T denotes the final time. What we look for in this study is what effects the added viscosity of the ADM model has on solutions at higher Reynold's numbers, when compared to those obtained using the stochastic Navier-Stokes equations. Specifically, we consider the Chorin flow example. We consider the flow flow in $D = (0.5, 1.5)\times(0.5, 1.5)$. The Reynolds number is $Re = 100$, the final time is $T = 1$, and the averaging radius is $\delta = h$. For the noise on the boundary we consider $K = 4$, $\omega_i \in [0,1], i = 1,2,3,4$ and the parabolic inflows $\Phi_1 =< 0, (x-1/2)(3/2-x) >$ on $\partial\Omega_1$, $\Phi_2 =< -(y-1/2)(3/2-y), 0 >$ on $\partial\Omega_2$, $\Phi_3 =< 0, -(x-1/2)(3/2-x) >$ on $\partial\Omega_3$, $\Phi_4 =< (y-1/2)(3/2-y), 0 >$ on $\partial\Omega_4$, where $\Omega_1 : 1/2 \le x \le 3/2, y = 1/2$, $\Omega_2 : 1/2 \le y \le 3/2, x = 3/2$, $\Omega_3 : 1/2 \le x \le 3/2, y = 3/2$, $\Omega_1 : 1/2 \le y \le 3/2, x = 1/2$.

5. Parameter identification and control of elliptic equations with random input data [12]

Let $(\Omega, \mathcal{F}, P)$ denote a complete probability space, where Ω is the set of outcomes, $\mathcal{F} \subset 2^{\Omega}$ is the σ-algebra of events and $P : \mathcal{F} \to [0,1]$ is a probability measure. Let $D \subset \mathbb{R}^d$ denote a bounded region. We consider the elliptic PDE

$$-\nabla\cdot(\kappa(\omega,x)\nabla u(\omega,x)) = f(\omega,x) \qquad \text{in } \Omega\times D \tag{9}$$

Table 1: *Expected value and variance of the difference between the direct numerical simulation and approximate deconvolution solutions for flow past a step;* error(E)= $\|\mathrm{E}[u_{ADM}(T) - u_{DNS}(T)]\|_{L^2(\Omega)}$ *and* error(var)= $\|\mathrm{var}[(u_{ADM} - u_{DNS}), \mathbf{x}, T]\|_{L^2(\Omega)}$.

h	error(E)	$rate(E)$	error(var)	$rate(var)$
1/4	0.0724682		0.00319198	
1/8	0.0297043	1.287	0.000543116	2.555
1/16	0.00947135	1.649	0.0000580259	3.226
1/32	0.00259538	1.868	0.00000437377	3.956
1/64	0.00065426	**1.988**	0.0000002741196	**3.996**

along with boundary conditions, where κ and f are random fields so that the solution u is random as well.

Given a random field κ, we formulate an *optimal control* problem associated with this SPDE: minimize the cost functional

$$E\big[\|u(\omega,\cdot) - \overline{u}(\omega,\cdot)\|^2_{L^2(D)} + \|f(\omega,\cdot)\|^2_{L^2(D)}\big] \tag{10}$$

on all $u \in L^2_P(\Omega; H^1_0(D) \cap H^2(D))$ and $f \in L^2_P(\Omega, L^2(D))$, subject to (9), where $P(\omega \in \Omega| \quad \kappa_{\min} \leq \kappa(\omega, x) \leq \kappa_{\max}, \forall x \in D) = 1$ for some $\kappa_{\min}, \kappa_{\max} > 0$. The solution to (9) must be understood in a variational sense. Using standard techniques one can prove that the problem (9)–(10) has a unique optimal pair $(\widehat{u}, \widehat{f}) \in L^2_P(\Omega; H^1_0(D)) \times L^2_P(\Omega, L^2(D))$ that is characterized by a maximum principle type result. The following results shows that the optimal pair may be found by solving an optimality system.

Theorem 1 $(\widehat{u}, \widehat{f})$ *is the unique optimal pair in problem (9)–(10) if and only if there exists* $\xi \in L^2_P(\Omega; H^1_0(D))$ *that satisifies (9) and*

$$-\nabla \cdot (\kappa(\omega, x)\nabla\xi(\omega, x)) \quad = \widehat{u}(\omega, x) - \overline{u}(\omega, x) \quad \text{in } \Omega \times D \tag{11}$$

$$\widehat{f}(\omega, x) = -\frac{1}{\alpha}\, m(x)\xi(\omega, x) \qquad \text{a.e. in } \Omega \times D. \tag{12}$$

The second problem we consider is the *identification of the coefficient* κ in the SPDE (9), see e.g. [**?**, 14]. We are given a possible random observation $\overline{u}$ corresponding to the state variable u and we must determine κ in (9) such that $u(\kappa) = u$ in $\Omega \times D$. Of course, such a κ may not exist. The least squares approach leads us to the minimization problem: for $\overline{u} \in L^2_P(\Omega)$ given and on all $u \in L^2_P(\Omega; H^1_0(D))$ and $\kappa \in L^2_p(\Omega; L^2(D))$,

$$\text{Minimize} \quad E\big[\|u - \overline{u}\|^2_{L^2(D)} + \beta\|\kappa\|^2_{L^2(D)}\big] \qquad \text{subject to (9).} \tag{13}$$

We again obtain solutions of this problem from an optimality system.

Theorem 2 *Let* (u^*, κ^*) *be an optimal pair in problem (9) and (13). Then*

$$\kappa^*(\omega, x) = -\frac{1}{\beta}\nabla u^*(\omega, x)\nabla\xi(\omega, x) \qquad \text{a.e. in } \Omega \times D, \tag{14}$$

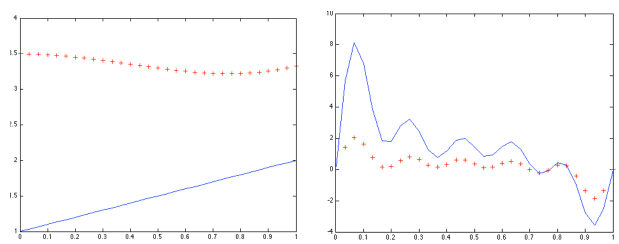
where $\xi \in L^2_P(\Omega; H^1_0(D))$ *is the solution of (11).*

We consider approximations to the random fields κ, $\overline{u}$, and f determined from truncated Karhunen-Loève expansions. Assuming a finite dimensional truncation of the noise, we turn the original control problem for a stochastic elliptic equation into a deterministic parametric elliptic system and use finite element techniques to approximate the solution of the resulting deterministic problem ([17, 21]. In this preliminary effort, we follow the work [22] and seek a numerical approximation to the exact solution of the discrete optimality system in a piecewise linear finite dimensional space coupled with anisotropic Smolyak collocation points. We apply a gradient algorithm to determine the optimal pair (u^*, κ^*).

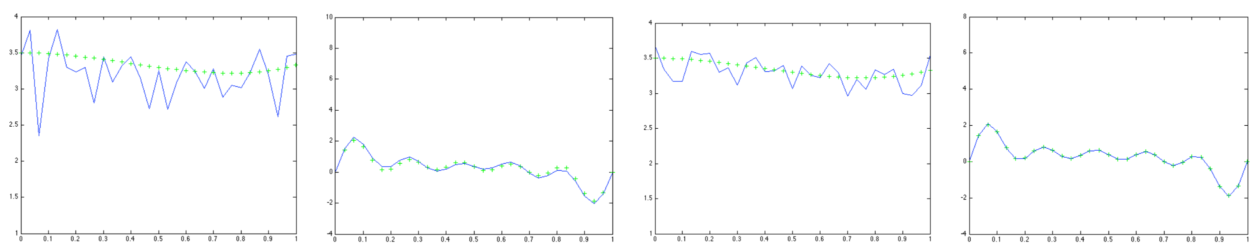
We present representative computational results for the parameter identification problem having the exact solution

$$\overline{u} = u^* = x(1-x^2) + \sum_{n=1}^{N} \sin\left(\frac{m\pi x}{L}\right) Y_n(\omega) \qquad \text{and} \qquad \kappa^* = (1+x^3) + \sum_{n=1}^{N} \cos\left(\frac{n\pi x}{L}\right) Y_n(\omega)$$

from which we determine that $f = \nabla \cdot (\kappa^*(\omega, x)\nabla\overline{u}(\omega, x))$. We also choose $\kappa = 1 + x$ as the initial guess for the optimizer. The figures that follow show the typical behavior of the computational experiments in which the greater efficiency gained by using anisotropic Smolyak collocation points is demonstrated.



5 random variables case. Expected value of initial (solid line) and target (+) coefficient κ (left) and u (right).



5 random variables and 11 samples case. Expected value of optimal and target coefficient κ (left and middle right) and of optimal and target solution u (middle left and right). Left two plots are for Monte Carlo and the right two plots are for anisotropic Smolyak sampling.

That superiority is also illustrated in the table that follows.

N	AS	MC	N	AS	MC	N	AS	MC
5	801	7e+03	10	1581	9e+06	20	11561	8e+09

With $N = 5$, 10 and 20 random variables, the number of deterministic solutions required by anisotropic Smolyak collocation (AS) based on Clenshaw-Curtis quadrature rules and the Monte Carlo (MC) method using random samples to reduce the original error in both $\mathbb{E}\left[\|u-\overline{u}\|_{L^2(D)}\right]$ and $\mathbb{E}\left[\|\kappa-\kappa^*\|_{L^2(D)}\right]$ by a factor of 10^6.

6. Nonlinear spectral eddy-viscosity models of turbulence [8]

Algebraic turbulence models, i.e., models in which an added viscosity is introcuded that is algebraically related to the stress tensor or the velocity gradient, are among the simplest ones developed. The most common such model was developed independently by Ladyzhenskaya [20] and Smagorinsky [23] in which the added *eddy viscosity* ν_t is proportional to $|\nabla u|^p$, where u denotes the velocity; the Smagorinsky case is $p = 1$. This type of model has been found to be over-difussive, i.e., not only are small-scale behaviors damped, but large-scale, i.e., low-frequency behaviors, are damped as well. This is one reason, perhaps the main one, why so many other, more complex, turbulence models have been invented, including ones that require the solution of additional nonlinear partial differential equations.

Our approach is to return to the Ladyzhenskaya/Smagorinsky model, but to apply a high-pass filter Q_M to the the added viscosity term so that, in the incompressible flow case, the momentum equation for the turbulence model looks like

$$\rho(u_t + u \cdot \nabla u) + \nabla p = \nu \Delta u + \epsilon Q_M \nabla \cdot \big(|Q_M \nabla u|^p \nabla Q_M u \big),$$

where p denotes the pressure, ρ the constant density, Δ the Laplace operator, and ϵ a modeling parameter. If Q_M is the identity operator, this equation reduces to the Ladyzhenskaya/Smagorinsky model. In this way, the additional viscosity is introduced only at the small, i.e., high-frequency, scales. This is, of course, not a new idea. For hyperbolic conservation laws, this is exactly the idea of the spectral viscosity method developed by Tadmor and co-workers in [25] and several related papers and later extended to the finite element and wavelet cases [15, 16, 18]. In the Navier-Stokes setting, a two-grid implementation was effected in [19] where promising results were reported on.

In a spectral method setting, the high-pass filter is easy to apply: Q_M simply cuts out any frequency lower than M. In the finite element setting, things are somewhat more complicated because standard finite element basis functions all operate at the same frequency. For this reason, hierarchical basis function are used; such a basis has frequency levels as opposed to individual frequencies, but that suffices to define a high-pass filter.

We have analyzed the well-posedness and convergence of a spectral method implementation of this approach in the forthcoming paper [**?**]; in that work, an optimal relation between the parameters M, p, and ϵ is given. We will soon do likewise for the hierarchical finite element case. In the next year, we will carry out the comparisons between statistical information obtained using stochastic versions of theses models, compared to the analogous information obtained using the stochastic Navier-Stokes equations.

7. Brief descriptions of other work

ANOVA expansions and efficient sampling methods for parameter dependent nonlinear PDEs [1]

The impact of parameter dependent boundary conditions on solutions of a class of nonlinear partial differential equations and on optimization problems constrained by such equations is considered. The tools used to gain insights about these issues are the Analysis of Variance (ANOVA) expansion of functions and the related notion of the effective dimension of a function; both concepts are reviewed. The effective dimension is then used to study the accuracy of truncated ANOVA expansions. Then, based on the ANOVA expansions of functionals of the solutions, the effects of different parameter sampling methods on the accuracy of surrogate optimization approaches to constrained optimization problems are considered. Demonstrations are given to show that whenever truncated ANOVA expansions of functionals provide accurate approximations, optimizers found through a simple surrogate optimization strategy are also relatively accurate. Although the results are presented and discussed in the context of surrogate optimization problems, the results also apply to other settings such as stochastic ensemble methods and reduced-order modeling for nonlinear partial differential equations.

Error analysis of finite element approximations of the stochastic Stokes equations [2]

Numerical solutions of the stochastic Stokes equations driven by white noise perturbed forcing terms using finite element methods are considered. The discretization of the white noise and finite element approximation algorithms are studied. The rate of convergence of the finite element approximations is proved to be almost first order ($h|\ln h|$) in two dimensions and one half order ($h^{1/2}$) in three dimensions. Numerical results using the algorithms developed are also presented.

Error estimates of stochastic optimal Neumann boundary control problems [9]

We study mathematically and computationally optimal control problems for stochastic partial differential equations with Neumann boundary conditions. The control objective is to minimize the expectation of a cost functional, and the control is of the deterministic, boundary-value type. Mathematically, we prove the existence of an optimal solution and of a Lagrange multiplier; we represent the input data in terms of their Karhunen-Loève expansions and deduce the deterministic optimality system of equations. Computationally, we approximate use a finite element discretization of the optimality system and estimate its error with respect to both spatial and random parameter spaces.

Option pricing in the presence of random arbitrage return [3]

We consider option pricing problems when we relax the condition of no arbitrage in the Black-Scholes model. Assuming random noise in the interest rate process, the derived pricing equation is in the form of stochastic partial differential equation. We use a Karhunen-Loève expansion to approximate the stochastic term and a combined finite difference/finite element method to effect temporal and "spatial" discretization. Computational examples in which the noise is assumed to be a OrnsteinUhlenbeck process are provided that illustrate not only the discretization methods used, but the type of results relevant to option pricing that can be obtained from the model.

High accuracy method for turbulent flow problems [6]

We present a method of high-order temporal and spatial accuracy for flow problems with high Reynolds number. The method presented is stable, computationally cheap, and gives an accurate approximation to the quantities sought. The direct numerical simulation of turbulent flows is computationally expensive or not even feasible. Hence, the method employs turbulence modeling. The two key ingredients are the deferred correction method combined with the family of approximate deconvolution models, which allows for arbitrarily high order of accuracy. We prove stability and accuracy for the two-step method; the method is shown to be second-order accurate in time and in the filtering width.

Modeling and analyses of thermal fluctuations and inhomogeneities in materials [4]

To gain a better understanding of such effects, the Ginzburg-Landau model for superconductivity is studied in situations for which the described physical processes are subject to uncertainty, modeled by SPDEs with additive and multiplicative white noise. The existence and uniqueness of weak and strong statistical solutions are proved.

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Publications

See publications [1]– [13] in the above reference list.

Honors & Awards Received

Max Gunzburger: SIAM, Charter Fellow of the Society, 2009
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